

Wednesday, October 22, Lecture 23: Applications
of Residue Theory

Let $f: \Omega \rightarrow \mathbb{C}$ holomorphic except at a finite set of points $z_1, \dots, z_N$ inside γ . Then

$$\int_{\gamma} f(z) dz = 2\pi i \sum_{k=1}^N \operatorname{Res}_{z_k} f.$$



Main application of Residue Thm: Crucial integrals!
 Many integrals in physics and engineering cannot be computed without complex analysis.

Ex: Compute $I = \int_0^{2\pi} \frac{1}{5+4\cos\theta} d\theta$ (these sorts of integrals show up when studying pendulums and other oscillators)

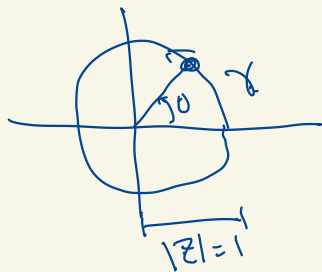
Soln:

Let

$$z = e^{i\theta} \quad (\text{so } \bar{z}(\theta) = \gamma(\theta) = e^{-i\theta}, \theta \in [0, 2\pi])$$

$$dz = ie^{i\theta} d\theta = iz d\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2}$$



$$\Rightarrow I = \int_{\gamma} \left(\frac{1}{5+2(z+\frac{1}{z})} \right) \frac{1}{iz} dz$$

$$= \frac{1}{i} \int_{\gamma} \frac{1}{(2z+1)(z+2)} dz$$

Residue Thm: f has poles at $z = -\frac{1}{2}$ and $z = -2$



$$\text{so } iI = 2\pi i \operatorname{Res}_{-\frac{1}{2}} f$$

- Since $-\frac{1}{2}$ is a simple pole,

$$\operatorname{Res}_{-\frac{1}{2}} f = \lim_{z \rightarrow -\frac{1}{2}} (z + \frac{1}{2}) \cdot f = \frac{1}{2(z+2)} \Big|_{z=-\frac{1}{2}} = \frac{1}{3}$$

$$\text{so } I = 2\pi \left(\frac{1}{3} \right) = \frac{2\pi}{3}$$

- Not too hard to see this works for other rational functions of θ !

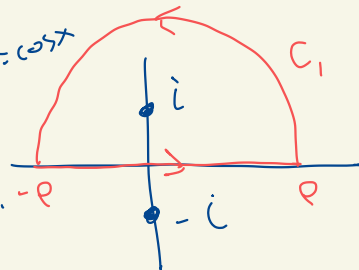
Ex: (First Fourier mode of rational function)

Evaluate $I = \int_0^\infty \frac{\cos x}{1+x^2} dx$.

Soln:

Observe that $e^{ix} = \cos x + i \sin x$, so $\operatorname{Re}(e^{ix}) = \cos x$

$f(z) = \frac{e^{iz}}{(z+i)(z-i)}$, both simple poles.



So compute, by Residue thm

$$2\pi i \operatorname{Res}_i f = \int_{-R}^R f dz + \int_{C_1} f dz$$

$$\downarrow$$

$$= \lim_{z \rightarrow i} (z-i) f = \lim_{z \rightarrow i} \frac{e^{iz}}{z-i} = \frac{e^{-1}}{2i}$$

$$\int_{C_1} |f| dz = \int_C \left| \frac{e^{iz}}{z^2+1} \right| dz$$

$e^{iz} = e^{i(x+iy)} = e^{ix} e^{-y}$

$$= \int_C \underbrace{|e^{ix}|}_{=1} \cdot \underbrace{|e^{-y}|}_{=e^{-y} < 1} \cdot \left| \frac{1}{z^2+1} \right| dz$$

$y > 0$.

$$= \int_C \left| \frac{1}{z^2+1} \right| dz$$

$$= \int_0^\pi \left| \frac{1}{(Re^{i\theta})^2+1} \right| i Re^{i\theta} d\theta$$

$$\leq \frac{R}{R^2} \cdot \pi \xrightarrow{R \rightarrow \infty} 0$$

So,

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1} dx = 2\pi i \left(\frac{1}{2ie} \right) = \frac{\pi}{e}$$

Taking the Real part of both sides,

$$= \int_{-\infty}^{\infty} \frac{\cos x}{x^2+1} dx = 2 \int_0^{\infty} \frac{\cos x}{x^2+1} dx$$

But this is an even function so

$$\Rightarrow I = \int_0^{\infty} \frac{\cos x}{x^2+1} dx = \frac{\pi}{2e}.$$

Not too hard to see that many functions of the form $\frac{p(x)}{q(x)}$ can be similarly handled...

... as long as they are integrable, anyways.

Remember, $\int_1^{\infty} \frac{1}{x} dx = \infty$, so many rational functions are not integrable.

Ex:

Compute

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx$$

This is closely related to Fourier transform of , which is a filter in signal processing.
(like, remove high frequencies).

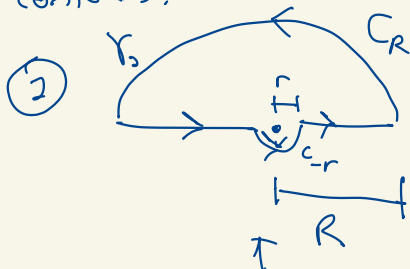
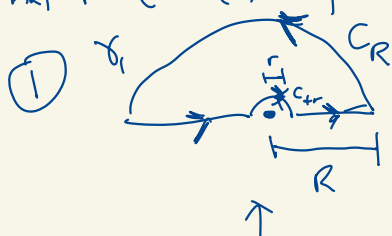
Soln:

Since $e^{ix} = \cos x + i \sin x$, we consider

$$\int \frac{e^{iz}}{z} dz$$

But there is a pole on the axis!

Notice that there are two plausible contours:



Residue thm:

$$\int_{\gamma_1} \frac{e^{iz}}{z} dz = 0$$

$$\int_{\gamma_2} \frac{e^{iz}}{z} dz = 2\pi i \operatorname{Res}_0 \frac{e^{iz}}{z}$$

$$\begin{aligned} \frac{e^{iz}}{z} &= \frac{1}{z} \left(1 + z + \frac{z^2}{2!} + \dots \right) \\ &= \frac{1}{z} + 1 + \frac{z}{2!} + \dots \\ \Rightarrow \operatorname{Res}_0 \frac{e^{iz}}{z} &= 1. \end{aligned}$$

So, these limits will disagree as $r \rightarrow 0$. This is because

$$\begin{aligned} \int_{C_{+r}} dz &= i\pi \\ \int_{C_{-r}} dz &= -i\pi \end{aligned}$$

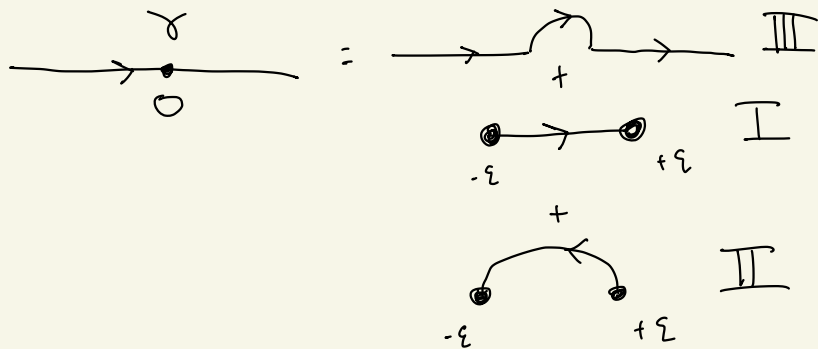
How to resolve?

Cauchy Principal Value: (often implicit in physics)
Means take the integral symmetrically:

Ex: $\int_{-3}^2 \frac{1}{x} dx$ does not converge.

$$\begin{aligned} \text{But } \text{p.v.} \int_{-3}^2 \frac{1}{x} dx &:= \lim_{\varepsilon \rightarrow 0^+} \left(\int_{-3}^{-\varepsilon} \frac{1}{x} dx + \int_{\varepsilon}^2 \frac{1}{x} dx \right) \\ &= \lim_{\varepsilon \rightarrow 0^+} \left(\log|x| \Big|_{-3}^{-\varepsilon} + \log|x| \Big|_{\varepsilon}^2 \right) \\ &= \lim_{\varepsilon \rightarrow 0^+} \left(\log \varepsilon - \log 3 + \log 2 - \log \varepsilon \right) \\ &= -\log 3 + \log 2 \end{aligned}$$

Let's use this.



$$\text{p.v.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = \lim_{\varepsilon \rightarrow 0} \left(\int_{\text{III}} \frac{e^{ix}}{x} dx + \int_{\text{I}} \frac{e^{ix}}{x} dx + \int_{\text{II}} \frac{e^{ix}}{x} dx \right)$$

Note: this p.v. detour is necessary because

$$\frac{e^{ix}}{x} = \frac{\cos x}{x} + i \frac{\sin x}{x}$$

So, $\text{Im}\left(\frac{e^{ix}}{x}\right) = \frac{\sin x}{x}$ is well-behaved on $\mathbb{R}_1$
but $\text{Re}\left(\frac{e^{ix}}{x}\right) = \frac{\cos x}{x}$ is not!

I

$$\int_I \frac{e^{iz}}{z} dz = \int_{-\xi}^{\xi} \frac{e^{ix}}{x} dx.$$

$$\text{But we want } \operatorname{Im} \left(\int_{-\xi}^{\xi} \frac{e^{ix}}{x} dx \right) = \int_{-\xi}^{\xi} \frac{\sin x}{x} dx.$$

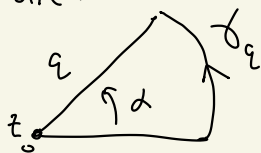
But $f(x) = \frac{\sin x}{x}$ $x \neq 0$, $f(0) = 1$ is continuous, so bounded:

$$\lim_{\xi \rightarrow 0} \int_{-\xi}^{\xi} \left| \frac{\sin x}{x} \right| dx \leq \lim_{\xi \rightarrow 0} M \int_{-\xi}^{\xi} dx = \lim_{\xi \rightarrow 0} M 2\xi \rightarrow 0.$$

II

Lemma:

Let $f(z)$ have a simple pole at z_0 and let γ_{ξ} be a circular arc:



Then

$$\lim_{\xi \rightarrow 0} \int_{\gamma_{\xi}} f(z) = \alpha i \operatorname{Res}_{z_0} f.$$

pf:

Locally $f(z) = \frac{a_{-1}}{z - z_0} + g(z)$. so $\lim_{\xi \rightarrow 0} = 0$ since g is holo.

$$\int_{\gamma_{\xi}} f = \int_0^{\alpha} \frac{a_{-1}}{\xi e^{i\theta}} \xi i e^{i\theta} d\theta + \int_0^{\alpha} g(\xi e^{i\theta}) \xi i e^{i\theta} d\theta = \alpha i a_{-1}.$$

So

$$\oint_{\Gamma} \frac{e^{iz}}{z} dz = \pi i \operatorname{Res}_{z=0} \frac{e^{iz}}{z} = \pi i.$$

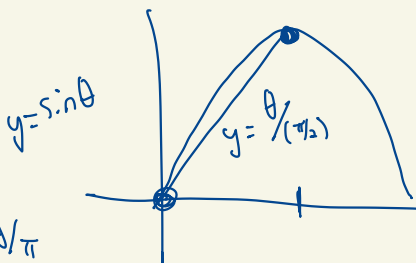
(III) We will need a tool for this part.

Jordan's lemma:

$$\int_0^{\pi} e^{-R \sin \theta} d\theta \leq \frac{\pi}{R}$$

pf:

on $[0, \pi/2]$,



so $e^{-R \sin \theta} \leq e^{-2R\theta/\pi}$ on $\theta \in [0, \pi/2]$ and

$$\int_0^{\pi/2} e^{-R \sin \theta} d\theta \leq \int_0^{\pi/2} e^{-2R\theta/\pi} d\theta$$

$$= \frac{\pi}{2R} (1 - e^{-R})$$

$$\leq \frac{\pi}{2R}.$$

By symmetry, we arrive at Jordan's lemma.

Now: Cauchy's thm + Jordan's lemma.

$$\int_{-\infty}^{\infty} f(z) dz = \oint_0 + \int_{C_R} f(z) dz$$

Estimate by Jordan's lemma:

So

$$\left| - \int_{C_R} \frac{e^{iz}}{z} dz \right| \leq \int_{C_R} \left| \frac{e^{iz}}{z} dz \right|$$

$$= \int_0^\pi \left| \frac{e^{iR(\cos\theta + i\sin\theta)}}{Re^{i\theta}} iRe^{i\theta} d\theta \right|$$

$$= \int_0^\pi \frac{R}{R} |e^{iR\cos\theta} e^{-R\sin\theta}| d\theta$$

$$= \int_0^\pi e^{-R\sin\theta} d\theta$$

Jordan's lemma

$$\rightarrow \leq \frac{\pi}{R} \xrightarrow{R \rightarrow \infty} 0$$

Which forces $\int_{\mathbb{R}} f(z) dz = 0$.

Put it all together:

Since $\frac{\sin x}{x}$ is continuous, it is integrable and so

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \int_{-\infty}^{\infty} \frac{\sin x}{x} dx$$

$$\operatorname{Im} \left(\text{p.v.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx \right) = \operatorname{Im} \left(\lim_{\epsilon \rightarrow 0^+} \int_{\text{I} + \text{II} + \text{III}} \frac{e^{iz}}{z} dz \right)$$

$$= 0 + 0 + \operatorname{Im}(\pi i)$$

So

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi.$$